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COPLANAR MOTION OF TWO PLANETS, ONE HAVING A ZERO MASS.

By Dr. G. W. HILL, Washington, D. C.

The supposition that two planets circulate about their central body in the same plane enables us to dispense with two differential equations of the second order in the general problem of three bodies. The further supposition, that the mass of one of them is too insignificant to have any sensible effect on the motion of the other, enables us to consider the motion of the latter as known and as taking place according to the laws of Kepler. Hence, in this case, the two co-ordinates of the planet of zero mass are the only unknowns; and they are given by two differential equations of the second order. These suppositions have, approximately, place in several cases in the solar system, but I have more especially in view the motion of the satellite Hyperion as disturbed by the action of Titan. My object in this paper is simply to point out a method of proceeding, which may, I think, be advantageously employed in this case.

Employing the usual notation x, y, r , for the rectangular co-ordinates and radius vector of the planet whose motion is to be determined, x', y', r' , for the corresponding quantities belonging to the acting planet, m' the mass of the latter, and M the mass of the central body, the differential equations of motion will be

$$\frac{d^2x}{dt^2} = \frac{\partial Q}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial Q}{\partial y},$$

where Q , the potential function, has the following expression:—

$$Q = \frac{M}{\sqrt{(x^2 + y^2)}} + m' \left[\frac{1}{\sqrt{[(x - x')^2 + (y - y')^2]}} - \frac{x'x + y'y}{r'^3} \right].$$

The co-ordinates of m' satisfy the differential equations

$$\frac{d^2x'}{dt^2} + \frac{M + m'}{r'^3} x' = 0, \quad \frac{d^2y'}{dt^2} + \frac{M + m'}{r'^3} y' = 0.$$

We can, without any loss of generality, assume that the axis of x is directed toward the lower apsis of m' . Then the integrals of the last-stated differential equations are

$$x' = a' (\cos \varepsilon' - e'), \quad y' = a' \sqrt{1 - e'^2} \sin \varepsilon',$$

where ε' is derived from the equation

$$n't + c' = \varepsilon' - e' \sin \varepsilon',$$

a' , e' , c' being constants, and n' being the equivalent of $\sqrt{\left(\frac{M+m'}{a'^3}\right)}$.

It is desirable to know what the differential equations determining x and y become when expressed in terms of any other variables. For this end Lagrange's canonical form of the equations serves very conveniently. Let the new variables be u and s , and employ the subscript (1) to denote the complete differential coefficient with respect to t of any variable to which it is attached. Then T standing for $\frac{1}{2}(x_1^2 + y_1^2)$ expressed in terms of u , s , u_1 , s_1 , Lagrange's canonical form of the equations is

$$\frac{d}{dt} \frac{\partial T}{\partial u_1} - \frac{\partial T}{\partial u} = \frac{\partial Q}{\partial u}, \quad \frac{d}{dt} \frac{\partial T}{\partial s_1} - \frac{\partial T}{\partial s} = \frac{\partial Q}{\partial s}.$$

As we have

$$x_1 = \frac{\partial x}{\partial u} u_1 + \frac{\partial x}{\partial s} s_1 + \frac{\partial x}{\partial t},$$

$$y_1 = \frac{\partial y}{\partial u} u_1 + \frac{\partial y}{\partial s} s_1 + \frac{\partial y}{\partial t},$$

we get

$$\begin{aligned} T = & \frac{1}{2} \left[\left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 \right] u_1^2 + \left(\frac{\partial x}{\partial u} \frac{\partial x}{\partial s} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial s} \right) u_1 s_1 \\ & + \frac{1}{2} \left[\left(\frac{\partial x}{\partial s} \right)^2 + \left(\frac{\partial y}{\partial s} \right)^2 \right] s_1^2 + \left(\frac{\partial x}{\partial u} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial t} \right) u_1 \\ & + \left(\frac{\partial x}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial s} \frac{\partial y}{\partial t} \right) s_1 + \frac{1}{2} \left[\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right]. \end{aligned}$$

It is very plain from the form of Lagrange's equations that if the variables u and s were so assumed that one of them, u for instance, should disappear at once from the expressions for T and Q , we should have an integral of the problem. For then $\frac{d}{dt} \frac{\partial T}{\partial u_1} = 0$; and, integrating, $\frac{\partial T}{\partial u_1} = \text{a constant}$. This selection, in a theoretical sense, is always possible, and in as many essentially distinct ways as there are first integrals of the problem, which, in the present case, are four. But, although it is easy, in innumerable ways, to make Q depend on one variable, it is not so easy to make the six factors of the general expression for T depend solely on the same variable. And, when we inquire what equations must be satisfied for this, we find that they are essentially the same as those which are satisfied by the Eulerian multipliers. Hence, nothing is gained by approaching the problem from this side.

I propose to take u and s so that

$$x = \rho x' u + \rho y' s, \quad y = \rho y' u - \rho x' s,$$

where ρ denotes a function of t supposed known, but, for the present, left indeterminate. From these equations may be derived

$$r^2 = \rho^2 r'^2 (u^2 + s^2), \quad x'x + y'y = \rho r'^2 u.$$

Hence the potential function, in terms of u and s , becomes

$$\Omega = \frac{1}{\rho r'} \left[\frac{M}{\sqrt{(u^2 + s^2)}} + \frac{m'}{\sqrt{(u - \rho^{-1})^2 + s^2}} - m' \rho^2 u \right].$$

In the general expression for T we substitute the values

$$\begin{aligned} \frac{\partial x}{\partial u} &= \rho x', & \frac{\partial y}{\partial u} &= \rho y', & \frac{\partial x}{\partial s} &= \rho y', & \frac{\partial y}{\partial s} &= -\rho x', \\ \frac{\partial x}{\partial t} &= \frac{d(\rho x')}{dt} u + \frac{d(\rho y')}{dt} s, & \frac{\partial y}{\partial t} &= \frac{d(\rho y')}{dt} u - \frac{d(\rho x')}{dt} s. \end{aligned}$$

The result is

$$\begin{aligned} T &= \frac{1}{2} \rho^2 r'^2 (u_1^2 + s_1^2) - a'^2 n' \sqrt{(1 - e'^2)} \rho^2 (us_1 - su_1) \\ &\quad + \frac{1}{2} \left[a'^2 n'^2 \left(\frac{2a'}{r'} - 1 \right) \rho^2 + 2r' \frac{dr'}{dt} \rho \frac{d\rho}{dt} + r'^2 \frac{d\rho^2}{dt^2} \right] (u^2 + s^2). \end{aligned}$$

For the sake of brevity we may write, h_1, h_2, h_3 being known functions of t ,

$$T = \frac{1}{2} h_1 (u_1^2 + s_1^2) - h_2 (us_1 - su_1) + \frac{1}{2} h_3 (u^2 + s^2).$$

This, substituted in Lagrange's canonical form of the differential equations, gives as the equations of the problem,

$$\begin{aligned} \frac{d}{dt} \left(h_1 \frac{du}{dt} \right) + 2h_2 \frac{ds}{dt} - h_3 u + \frac{dh_2}{dt} s &= \frac{\partial \Omega}{\partial u}, \\ \frac{d}{dt} \left(h_1 \frac{ds}{dt} \right) - 2h_2 \frac{du}{dt} - \frac{dh_2}{dt} u - h_3 s &= \frac{\partial \Omega}{\partial s}. \end{aligned}$$

Let us now adopt a more general independent variable than the time. Calling this ζ , let $dt = \theta d\zeta$, in which θ may be regarded as a function either of t or ζ . The second supposition will be the more advantageous. In either case as we obtain, on integrating, u and s as functions of ζ , it will be necessary to have the values of ζ which correspond to given values of the time, and thus the inverse function will have to be considered. Then, in terms of the new independent variable,

$$\begin{aligned} \frac{d}{d\zeta} \left(\frac{h_1}{\theta} \frac{du}{d\zeta} \right) + 2h_2 \frac{ds}{d\zeta} - \theta h_3 u + \frac{dh_2}{d\zeta} s &= \frac{\partial (\theta \Omega)}{\partial u}, \\ \frac{d}{d\zeta} \left(\frac{h_1}{\theta} \frac{ds}{d\zeta} \right) - 2h_2 \frac{du}{d\zeta} - \frac{dh_2}{d\zeta} s - \theta h_3 u &= \frac{\partial (\theta \Omega)}{\partial s}. \end{aligned}$$

We can now consider how ρ and θ should be assumed in order that the differential equations may be most simplified. In the first place it appears impor-

tant that the potential function Ω should be freed from the independent variable ζ . This is accomplished by putting $\rho = 1$. In the second place it seems we cannot readily do better than take the eccentric anomaly ε' of the attracting planet as the independent variable ζ . Then

$$dt = \frac{r'}{a'n'} d\varepsilon', \quad \text{and} \quad \theta = \frac{r'}{a'n'}.$$

Also we have

$$h_1 / \theta = a'^2 n' (1 - e' \cos \varepsilon'), \quad h_2 = a'^2 n' \sqrt{1 - e'^2}, \quad \theta h_3 = a'^2 n' (1 + e' \cos \varepsilon'),$$

$$\frac{\theta}{\rho r'} = \frac{1}{a'n'}, \quad M + m' = a'^3 n'^2.$$

For the sake of simplicity let the signification of Ω be changed, and, putting $\frac{m'}{M+m'} = \nu$, let

$$\Omega = \frac{1-\nu}{\sqrt{(u^2+s^2)}} + \frac{\nu}{\sqrt{[(u-1)^2+s^2]}} - \nu u.$$

Then our differential equations take the form

$$\frac{d}{d\varepsilon'} \left[(1 - e' \cos \varepsilon') \frac{du}{d\varepsilon'} \right] + 2\sqrt{1 - e'^2} \frac{ds}{d\varepsilon'} - (1 + e' \cos \varepsilon') u = \frac{\partial \Omega}{\partial u},$$

$$\frac{d}{d\varepsilon'} \left[(1 - e' \cos \varepsilon') \frac{ds}{d\varepsilon'} \right] - 2\sqrt{1 - e'^2} \frac{du}{d\varepsilon'} - (1 + e' \cos \varepsilon') s = \frac{\partial \Omega}{\partial s}.$$

It will be noticed that the potential function Ω is, by this assumption of variables, completely freed from co-ordinates expressing the position of the attracting planet; and that the two factors $1 - e' \cos \varepsilon'$ and $1 + e' \cos \varepsilon'$, very simple functions of the independent variable ε' , are the only evidences of the position of this body in the differential equations. And, of the four elements of its orbit, e' is the only one we have to deal with.

We propose now to see whether the introduction of elliptic co-ordinates will bring about any simplification in the problem. Supposing

$$x_1 = s, \quad x_2 = u - \frac{1}{2},$$

let
$$\frac{x_1^2}{a_1 + \lambda_1} + \frac{x_2^2}{a_2 + \lambda_1} = 1, \quad \text{and} \quad \frac{x_1^2}{a_1 + \lambda_2} + \frac{x_2^2}{a_2 + \lambda_2} = 1,$$

be the equations of a confocal ellipse and hyperbola, a_1 and a_2 being constants, and λ_1 and λ_2 the new variables destined to take the place of u and s . By eliminating x_2^2 from these equations we obtain

$$\frac{a_1 - a_2}{(a_1 + \lambda_1)(a_1 + \lambda_2)} x_1^2 = 1;$$

whence

$$x_1 = \sqrt{\left[\frac{(a_1 + \lambda_1)(a_1 + \lambda_2)}{a_1 - a_2} \right]}.$$

The expression of x_2 in terms of λ_1 and λ_2 is obtained from this by simply interchanging a_1 and a_2 . Thus

$$x_2 = \sqrt{\left[\frac{(a_2 + \lambda_1)(a_2 + \lambda_2)}{a_2 - a_1} \right]}.$$

We now proceed to find what Ω becomes in terms of λ_1 and λ_2 . By taking the sum of the squares of the last two equations we get

$$x_1^2 + x_2^2 = a_1 + a_2 + \lambda_1 + \lambda_2.$$

Thus far a_1 and a_2 have been left indeterminate, but we now assume

$$a_2 - a_1 = \frac{1}{4}.$$

Then

$$\begin{aligned} u^2 + s^2 &= (x_2 + \tfrac{1}{2})^2 + x_1^2 \\ &= 2a_2 + \lambda_1 + \lambda_2 + 2\sqrt{[(a_2 + \lambda_1)(a_2 + \lambda_2)]} \\ &= [\sqrt{(a_2 + \lambda_1)} + \sqrt{(a_2 + \lambda_2)}]^2, \\ \sqrt{(u^2 + s^2)} &= \sqrt{(a_2 + \lambda_1)} + \sqrt{(a_2 + \lambda_2)}, \\ (u - 1)^2 + s^2 &= (x_2 - \tfrac{1}{2})^2 + x_1^2 \\ &= 2a_2 + \lambda_1 + \lambda_2 - 2\sqrt{[(a_2 + \lambda_1)(a_2 + \lambda_2)]}, \\ \sqrt{[(u - 1)^2 + s^2]} &= \sqrt{(a_2 + \lambda_1)} - \sqrt{(a_2 + \lambda_2)}, \\ u &= 2\sqrt{[(a_2 + \lambda_1)(a_2 + \lambda_2)]} + \tfrac{1}{2}. \end{aligned}$$

For the sake of brevity we will now put

$$\sqrt{(a_2 + \lambda_1)} = p, \quad \sqrt{(a_2 + \lambda_2)} = q.$$

Then it is plain Ω may be written

$$\begin{aligned} \Omega &= \frac{1 - \nu}{p + q} + \frac{\nu}{p - q} - 2\nu pq \\ &= \frac{1 - \nu}{p + q} + \frac{\nu}{p - q} - \tfrac{1}{2}\nu(p + q)^2 + \tfrac{1}{2}\nu(p - q)^2. \end{aligned}$$

We have now to deal with T . By taking the logarithms of the values of x_1^2 and x_2^2 , and then differentiating, we obtain

$$\begin{aligned} 2 \frac{dx_1}{x_1} &= \frac{d\lambda_1}{a_1 + \lambda_1} + \frac{d\lambda_2}{a_1 + \lambda_2}, \\ 2 \frac{dx_2}{x_2} &= \frac{d\lambda_1}{a_2 + \lambda_1} + \frac{d\lambda_2}{a_2 + \lambda_2}. \end{aligned}$$

Whence may be derived

$$4(dx_1^2 + dx_2^2) = \left[\frac{x_1^2}{(a_1 + \lambda_1)^2} + \frac{x_2^2}{(a_2 + \lambda_1)^2} \right] d\lambda_1^2 + \left[\frac{x_1^2}{(a_1 + \lambda_2)^2} + \frac{x_2^2}{(a_2 + \lambda_2)^2} \right] d\lambda_2^2 \\ + 2 \left[\frac{x_1^2}{(a_1 + \lambda_1)(a_1 + \lambda_2)} + \frac{x_2^2}{(a_2 + \lambda_1)(a_2 + \lambda_2)} \right] d\lambda_1 d\lambda_2.$$

On substituting in the factor of $d\lambda_1 d\lambda_2$ the values of x_1^2 and x_2^2 it vanishes, and the expression takes the form

$$4(dx_1^2 + dx_2^2) = \frac{\lambda_1 - \lambda_2}{(a_1 + \lambda_1)(a_2 + \lambda_1)} d\lambda_1^2 + \frac{\lambda_2 - \lambda_1}{(a_1 + \lambda_2)(a_2 + \lambda_2)} d\lambda_2^2.$$

Or, in terms of p and q , we have

$$du^2 + ds^2 = \frac{p^2 - q^2}{p^2 - \frac{1}{4}} dp^2 + \frac{q^2 - p^2}{q^2 - \frac{1}{4}} dq^2.$$

In like manner we get

$$uds - sdu = (p + q) \left[\sqrt{\left(\frac{1}{4} - q^2 \right)} dp - \sqrt{\left(\frac{p^2 - 1}{4} \right)} dq \right].$$

The former expression for T was

$$T = \frac{1}{2} (1 - e' \cos \varepsilon') \frac{du^2 + ds^2}{d\varepsilon'^2} - \sqrt{1 - e'^2} \frac{uds - sdu}{d\varepsilon'} + \frac{1}{2} (1 + e' \cos \varepsilon') (u^2 + s^2);$$

hence, if we abbreviate by putting

$$\sqrt{\left(\frac{1}{4} - q^2 \right)} = a,$$

$$T = \frac{1}{2} (1 - e' \cos \varepsilon') \left[(1 + a^2) \frac{dp^2}{d\varepsilon'^2} + \left(1 + \frac{1}{a^2} \right) \frac{dq^2}{d\varepsilon'^2} \right] \\ - \sqrt{1 - e'^2} (p + q) \left[a \frac{dp}{d\varepsilon'} - \frac{1}{a} \frac{dq}{d\varepsilon'} \right] + \frac{1}{2} (1 + e' \cos \varepsilon') (p + q)^2.$$

T and Ω are somewhat simplified if we adopt variables ρ and σ , such that

$$p + q = \rho, \quad p - q = \sigma.$$

Also, for the sake of brevity, put

$$\frac{1}{2} \left(a + \frac{1}{a} \right) = h, \quad \frac{1}{2} \left(a - \frac{1}{a} \right) = k.$$

Then we have

$$T = \frac{1}{2} (1 - e' \cos \varepsilon') \left[h^2 \left(\frac{d\rho^2}{d\varepsilon'^2} + \frac{d\sigma^2}{d\varepsilon'^2} \right) - 2hk \frac{d\rho}{d\varepsilon'} \frac{d\sigma}{d\varepsilon'} \right] \\ - \sqrt{1 - e'^2} \rho \left(k \frac{d\rho}{d\varepsilon'} + h \frac{d\sigma}{d\varepsilon'} \right) + \frac{1}{2} (1 + e' \cos \varepsilon') \rho^2, \\ \Omega = \frac{1 - \nu}{\rho} + \frac{\nu}{\sigma} - \frac{1}{2} \nu \rho^2 + \frac{1}{2} \nu \sigma^2.$$

By this transformation \mathcal{Q} is considerably simplified; but, as more than offsetting this, T is rendered complex. As the expression for u in terms of these variables is

$$u = \sqrt{\left[\frac{1 - (\rho - \sigma)^2}{(\rho + \sigma)^2 - 1} \right]},$$

it will be perceived that h and k are trigonometrical functions of the angles of the triangle whose sides are 1, ρ , and σ , which might have been anticipated from geometrical considerations. Thus it appears no advantage would result from the employment of elliptic co-ordinates.

Returning, therefore, to the quasi-rectangular co-ordinates u and s , it seems some advantage would be gained if we adopt a new system of co-ordinates u and s , such that the new system is expressed, in terms of the old, as follows:—

$$u = u + s \sqrt{-1}, \quad s = u - s \sqrt{-1}.$$

We can also adopt the trigonometrical exponential corresponding to the arc ε' as the independent variable. Calling this $\zeta = e^{\varepsilon' \sqrt{-1}}$, an operator D is adopted, equivalent to $\zeta \frac{d}{d\zeta}$, so that $D \cdot \zeta^i = i \zeta^i$.

In terms of the new variables, \mathcal{Q} has the expression

$$\mathcal{Q} = \frac{1 - \nu}{\sqrt{us}} + \frac{\nu}{\sqrt{(u-1)(s-1)}} - \frac{1}{2}\nu(u+s).$$

And the differential equations are

$$D \{ [1 - \frac{1}{2}\varepsilon'(\zeta + \zeta^{-1})] Du \} + 2 \sqrt{-1} (1 - \varepsilon'^2) Du + [1 + \frac{1}{2}\varepsilon'(\zeta + \zeta^{-1})] u = -2 \frac{\partial \mathcal{Q}}{\partial s},$$

$$D \{ [1 - \frac{1}{2}\varepsilon'(\zeta + \zeta^{-1})] Ds \} - 2 \sqrt{-1} (1 - \varepsilon'^2) Ds + [1 + \frac{1}{2}\varepsilon'(\zeta + \zeta^{-1})] s = -2 \frac{\partial \mathcal{Q}}{\partial u}.$$

Only one of these equations need be actually employed, as either can be obtained from the other by changing the sign of $\sqrt{-1}$. We have

$$-2 \frac{\partial \mathcal{Q}}{\partial s} = \frac{1 - \nu}{\sqrt{u} \cdot \sqrt{s^3}} + \frac{\nu}{(u-1) \cdot \sqrt{(s-1)^3}} + \nu,$$

$$-2 \frac{\partial \mathcal{Q}}{\partial u} = \frac{1 - \nu}{\sqrt{u^3} \cdot \sqrt{s}} + \frac{\nu}{(u-1)^3 \cdot \sqrt{(s-1)}} + \nu.$$

For the purpose of integrating these equations, we may adopt the method of indeterminate coefficients; and we may employ, as proper to represent the values of u and s , the infinite series

$$u = \sum a_{i,j,k} \zeta^{ic + je' + k},$$

$$s = \sum a_{i,j,k} \zeta^{-ic - je' - k}.$$

Here i, j , and k denote positive or negative integers, zero included; and the summation must be extended so as to include all values for i, j , or k from $-\infty$ to $+\infty$. The a and c, c' are constants and functions of the four quantities e', ν, a and e ; a and e being two of the four arbitrary constants introduced by integration. The two remaining arbitrary constants serve only to complete the two elementary arguments which belong to the attracted planet, and, in this method of integration, they can pass unnoticed.

If we suppose that the orbit of the attracting planet is circular, the differential equations reduce to the very simple form.

$$(D+1)^2 u = -2 \frac{\partial \Omega}{\partial s},$$

$$(D-1)^2 s = -2 \frac{\partial \Omega}{\partial u}.$$

And, in this case, an integral can be found. For multiplying the first by Ds , and the second by Du , the sum of the equations, thus multiplied, is an exact derivative. Integrating, we get

$$Du Ds + us + 2\Omega = 2C,$$

C being the arbitrary constant.

This integral equation may be combined with the differential equations in such a way that one of the terms, regarded as the most difficult of expression in a developed form, may be eliminated. For example, if this is taken to be the term $\frac{\nu}{\sqrt{[(u-1)(s-1)]}}$ of Ω , the equations serving to determine the a may be taken to be

$$(s-1) D(D+2)u + \frac{1}{2} Du Ds + \frac{1-\nu}{\sqrt{(us)^3}} u + \frac{3}{2} (u-\nu)(s-\nu) + C = 0,$$

$$(u-1) D(D-2)s + \frac{1}{2} Du Ds + \frac{1-\nu}{\sqrt{(us)^3}} s + \frac{3}{2} (u-\nu)(s-\nu) + C = 0,$$

in which the constant C is not identical with the former C . One of these equations suffices, as the other is a consequence of it. The difference of these equations is simpler than either of them, and may be of use. It is

$$D[(u-1)Ds - (s-1)Du - 2(u-1)(s-1)] = \frac{1-\nu}{\sqrt{(us)^3}}(u-s).$$

In attempting to derive periodic series for the co-ordinates of Hyperion, it appears to me that it will be easier, in the first instance, to assume that Titan describes a circular orbit. And, in the next place, to assume that the perturbations are periodic functions of the mean elongation of the two bodies. And, as it may very easily happen that the terms, depending on the second and higher powers of the disturbing force, may quite alter the values of the coefficients, it

will be well to employ the method of mechanical quadratures. Starting Hyperion from its line of conjunction with Titan, and at right angles to this line, with an assumed velocity, trace out its path until the elongation, between the two bodies, amounts to 180° . Then, if Hyperion is again moving at right angles to its radius vector, the velocity at the start has been rightly assumed. But if not, one makes another trial; and, by interpolating between the two results, a velocity is obtained which will more nearly bring about this condition. And continued repetition of these trials will enable us to discover, with all desired approximation, the velocity which fulfils this condition. When the path of Hyperion, corresponding to this velocity, has been traced out, it will be easy, by the well-known processes of mechanical quadratures, to assign the periodic series representing the co-ordinates of the satellite under the supposed conditions.

When this is done, corrections to the co-ordinates, proportional to the first power of the satellite's proper eccentricity, can be obtained by the integration of a linear differential equation. By comparison of these with observation an approximate value of this proper eccentricity will be obtained; a thing to be desired as we seem to know next to nothing about it at present. Also one will be enabled to decide whether the motion of the mean anomaly is more rapid than that of the mean longitude, as has been asserted, without sufficient reason as it seems to me.

As illustrating this point, suppose that our moon, instead of having an eccentricity about 0.055, had one about 0.001. Then the variation would be the prevailing inequality, and the moon would appear to be in perigee always about syzygies, and in apogee about quadratures. In consequence the perigee would appear to retrograde with reference to the sun as fast as the moon advances with reference to the same body. And yet the relation between the motion of the argument, denominated the mean anomaly, and the motion of the mean longitude, would be nearly the same as it is at present. But the position of the perisaturnium of Hyperion has been concluded from its observed shortest and longest *radii vectores*. This is allowable only when the inequality, called the equation of the centre, is the overpowering one.

After the terms, proportional to the first power of the eccentricity, have been obtained, those factored by the second, third, etc., powers, can be derived by integrating differential equations of the same character.

In applying the process of mechanical quadratures to the motion of Hyperion, one will meet the difficulty of the uncertain value of the mass of Titan. But this cannot be avoided; an assumption must be made, and the results afterwards corrected by comparison with observation.

ON LOGARITHMIC ERRORS.

By PROF. H. A. HOWE, Denver, Col.

Mr. R. S. Woodward has kindly sent me the results of his computations of the average errors of interpolated values, dependent upon first differences, where the interpolating factor has the values 0.1, 0.2, . . . 0.9. I have compared these theoretical values with the values given by a discussion of the 1000 examples mentioned in No. 6 of Vol. I of the ANNALS. The comparison is given below:

| Interpolating factor. | Theoretical errors. | Actual errors. | Difference, $T - A$. |
|-----------------------|---------------------|----------------|-----------------------|
| 0.1 | 0.320 | 0.338 | - 0.018 |
| 0.2 | 0.303 | 0.288 | + 0.015 |
| 0.3 | 0.304 | 0.321 | - 0.017 |
| 0.4 | 0.290 | 0.268 | + 0.022 |
| 0.5 | 0.333 | 0.324 | + 0.009 |
| 0.6 | 0.290 | 0.276 | + 0.014 |
| 0.7 | 0.304 | 0.321 | - 0.017 |
| 0.8 | 0.303 | 0.289 | + 0.014 |
| 0.9 | 0.320 | 0.347 | - 0.027 |

The agreement of the theory with the observations is quite close, for the errors are carried to thousandths of a unit of the last place of the logarithm tables employed. It is strange that the differences are all negative (except one) when the interpolating factor is odd, and all positive when the factor is even. Furthermore, those actual errors which should be identical theoretically agree very closely.

It appears that the table given on pp. 126-7 of Vol. I does not represent well either the probable or the average errors, for the separate values of x , the interpolating factor. Hence the approximate theory developed in my articles, while it gives fair results, when general averages are taken, is not to be trusted for separate values of x .

It is to be hoped that Mr. Woodward may find time to prosecute further his admirable researches in this direction, and apply his theory to a variety of problems.

ON THE FREE COOLING OF A HOMOGENEOUS SPHERE, OF INITIAL
UNIFORM TEMPERATURE, IN A MEDIUM WHICH MAINTAINS A
CONSTANT SURFACE TEMPERATURE.

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1. The problem of the cooling of a homogeneous sphere initially heated to a uniform temperature divides itself naturally into two cases, namely: first, that in which the dissipation of heat at the surface of the sphere goes on independently of the surrounding medium, and second, that in which the dissipation is conditioned by the surrounding medium. For brevity we have applied the phrase "free cooling" to designate the first case. Both cases have been stated by Fourier and subsequent writers, but their investigations are, so far as we know, confined almost wholly to the second and more complex case. Moreover, they have, for the most part, contemplated the subject from the standpoint of the pure analyst; they have been too much occupied with the difficulties of their splendid analysis to give much heed to the needs of the computer.

The most interesting example for application of the theory of a cooling sphere is presented by the earth. The hypothesis most commonly entertained is, that the whole mass of the earth was at a certain epoch heated to a uniform temperature, and is now slowly cooling by conduction without sensibly heating surrounding space. With the ultimate object in view of tracing out the consequences of this hypothesis, we have sought to express the solutions of the case of free cooling and that of conditioned cooling in such terms that the computer can, without undue labor, assign the temperature at any point within the sphere for any value of the time.

In the following pages the case of the free cooling subject to the restriction of a constant coefficient of diffusion is alone considered. How closely the conditions of this restricted case accord with those actually presented by the earth is a question which requires examination in its proper place. The present enquiry, however, is directed solely to the mathematical treatment of the problem, and the conditions are tacitly assumed to apply to the earth.

2. The law of cooling of a homogeneous sphere, initially heated to a uniform temperature and cooling in a medium which maintains a constant surface temperature, must evidently be such as to give the same temperature for all points equidistant from the centre of the sphere.

Let u_0 be the initial uniform excess of the temperature of the sphere over that of the surrounding medium. Let u be the excess of the temperature of any shell of radius r and thickness dr over the temperature of the surrounding me-

dium at any time t , after the initial epoch. Denote the radius of the external surface of the sphere by r_0 ; and let the coefficient of diffusion which is here assumed to be constant, be symbolized by a^2 . Then the function of u_0, u, r_0, r, t , and a^2 which expresses the law of cooling must be such as to satisfy the following partial differential equation:—

$$\frac{\partial (ru)}{\partial t} = a^2 \frac{\partial^2 (ru)}{\partial r^2}. \quad (1)$$

If we write the functional relation which is to be discovered in the form

$$ru = f(u_0, r_0, r, t, a^2),$$

we must also have

$$ru = 0 \quad \text{for } r = r_0, \quad (2)$$

$$ru = 0 \quad \text{for } r = 0, \quad (3)$$

$$ru = ru_0 \quad \text{for } t = 0, \quad (4)$$

$$ru = 0 \quad \text{for } t = \infty. \quad (5)$$

Without going through the steps essential to exclude other functions, we observe that conditions (1) and (2) are satisfied by the expression.

$$C_n e^{-a^2 (n\pi/r_0)^2 t} \sin n\pi r/r_0,$$

C_n being any constant, and n any integer. Hence, since every term of this form will satisfy (1) and (2), we may write

$$ru = \sum_{n=1}^{n=\infty} C_n e^{-a^2 (n\pi/r_0)^2 t} \sin n\pi r/r_0. \quad (6)$$

When $t = 0$, (6) becomes in accordance with (4),

$$ru_0 = \sum_{n=1}^{n=\infty} C_n \sin n\pi r/r_0. \quad (7)$$

To fulfill this condition we must have

$$u_0 \int_0^{r_0} r \sin n\pi \frac{r}{r_0} dr = C_n \int_0^{r_0} \sin^2 n\pi \frac{r}{r_0} dr. \quad (8)$$

If we put

$$\theta = \pi \frac{r}{r_0},$$

(8) becomes

$$\frac{r_0 u_0}{\pi} \int_0^\pi \theta \sin n\theta d\theta = C_n \int_0^\pi \sin^2 n\theta d\theta,$$

whence

$$\begin{aligned} C_n &= -\frac{2r_0 u_0}{\pi} \cdot \frac{\cos n\pi}{n} \\ &= \frac{2r_0 u_0}{\pi} \cdot \frac{(-1)^{n+1}}{n}. \end{aligned} \quad (9)$$

Substituting this value of C_n in (6), we find the following equation, which expresses the law of cooling sought, in its simplest form:—

$$ru = \frac{2r_0 u_0}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} e^{-a^2 (n\pi/r_0)^2 t} \sin n\pi \frac{r}{r_0} \quad (10)$$

$$= \frac{2r_0 u_0}{\pi} \left\{ \begin{aligned} &+ e^{-a^2 (\pi/r_0)^2 t} \sin \pi \frac{r}{r_0} \\ &- \frac{1}{2} e^{-a^2 (2\pi/r_0)^2 t} \sin 2\pi \frac{r}{r_0} \\ &+ \frac{1}{3} e^{-a^2 (3\pi/r_0)^2 t} \sin 3\pi \frac{r}{r_0} \\ &- \dots \dots \dots \end{aligned} \right\}.$$

The equation (10) satisfies all of the conditions (1) to (5), because each term satisfies them.

When $r = 0$, (10) gives for the temperature at the centre of the sphere,

$$u = \frac{2r_0 u_0}{\pi} \cdot \frac{0}{0} = \frac{2r_0 u_0}{\pi} \cdot \frac{d(ru)}{dr}.$$

Evaluating this expression, we find for the temperature at the centre of the sphere,

$$u = 2u_0 \left[e^{-a^2 (\pi/r_0)^2 t} - e^{-a^2 (2\pi/r_0)^2 t} + e^{-a^2 (3\pi/r_0)^2 t} - \dots \right], \quad (11)$$

$r = 0.$

When $t = 0$ this gives $u = u_0$, as may be readily seen by recurring to (10). When $t > 0$ it appears that $u < u_0$.

3. The series (10) and (11) express all the circumstances of the cooling sphere under the assumed conditions. They are rapidly converging series when $a^2(\pi/r_0)^2 t$ is not much less than unity, since the negative exponents in the successive terms increase as the squares of the natural numbers. But in the most important application of this theory, namely, to the earth, the value of $a^2(\pi/r_0)^2 t$ is very small unless the time, t , is very large. Thus, for the earth, if the British foot and the year are the space and time units, we have in round numbers

$$a^2 = 400,*$$

$$r_0 = 21\,000\,000 \text{ feet.}$$

Hence, in order to make $a^2(\pi/r_0)^2 t$ unity, we must have

$$t = \frac{(21\,000\,000)^2}{400\pi^2},$$

* See section (c) of Sir William Thompson's paper "On the Secular Cooling of the Earth." Thompson and Tait's *Natural Philosophy*, Vol. I. Part II. Appendix (D). The coefficient refers to the unit of bulk of the substance of the earth's crust.

or about 100,000,000,000 years. When applied to the case of the earth, therefore, the series (10) and (11) are too slowly converging for the purposes of computation except when the time is very great.

To overcome this practical difficulty and obtain an expression which will readily give the temperature u , of any point of the sphere at any time, we proceed to transform the series (10.)

4. In the transformation which follows, and again in the sequel, we shall make use of a certain Eulerian integral or Gamma function whose value may here be briefly demonstrated, the reader being referred for a fuller investigation to the better treatises on the integral calculus. The integral is

$$v = \int_0^{\infty} e^{-a^2 x^2} dx \cos \beta x = \frac{\sqrt{\pi}}{2a} e^{-(\frac{1}{2}\beta/a)^2}.$$

To prove the equality of the second and third members of this equation, differentiate with respect to β and then integrate by parts. Thus

$$\begin{aligned} \frac{\partial v}{\partial \beta} &= - \int_0^{\infty} e^{-a^2 x^2} x dx \sin \beta x \\ &= \left[\frac{e^{-a^2 x^2} \sin \beta x}{2a^2} \right]_0^{\infty} - \frac{\beta}{2a^2} \int_0^{\infty} e^{-a^2 x^2} dx \cos \beta x. \end{aligned}$$

That is

$$\frac{\partial v}{\partial \beta} = - \frac{v\beta}{2a^2},$$

or

$$\frac{\partial v}{v} = - \frac{\beta \partial \beta}{2a^2};$$

whence

$$\log v = - \frac{\beta^2}{4a^2} + \log v_0,$$

or

$$v = v_0 e^{-(\frac{1}{2}\beta/a)^2}.$$

To determine the constant v_0 we observe that when $\beta = 0$,

$$v = v_0 = \int_0^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a}.$$

Therefore

$$v = \frac{\sqrt{\pi}}{2a} e^{-(\frac{1}{2}\beta/a)^2} \quad \text{and} \quad e^{-(\frac{1}{2}\beta/a)^2} = \frac{2a}{\sqrt{\pi}} \int_0^{\infty} e^{-a^2 x^2} dx \cos \beta x.$$

5. Applying the preceding integral to the n^{th} term of equation (10), putting

$$\beta^2 = n^2 \kappa^2 \quad \text{and} \quad 4a^2 = r_0^2 / (a^2 t),$$

we have

$$e^{-a^2 (n\pi/r_0)^2 t} = e^{-n^2 \pi^2 / (r_0^2 / a^2 t)} = \frac{r_0}{a\sqrt{\pi t}} \int_0^\infty e^{-r_0^2 x^2 / (4a^2 t)} dx \cos n\pi x.$$

Hence equation (10) may be written

$$ru = \frac{2r_0^2 u_0}{a\pi\sqrt{\pi t}} \int_0^\infty e^{-r_0^2 x^2 / (4a^2 t)} dx \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} \cos nx \sin n\pi \frac{r}{r_0}. \quad (12)$$

Now, for brevity, put

$$x = y/\pi \quad \text{and} \quad \theta = \pi r/r_0.$$

Then (12) becomes

$$\begin{aligned} ru &= \frac{2r_0^2 u_0}{a\pi\sqrt{\pi t}} \int_0^\infty e^{-r_0^2 y^2 / (4a^2 \pi^2 t)} dy \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} \cos ny \sin n\theta \\ &= \frac{r_0^2 u_0}{a\pi^2 \sqrt{\pi t}} \int_0^\infty e^{-r_0^2 y^2 / (4a^2 \pi^2 t)} dy \left\{ \begin{aligned} &+ \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} \sin n(y + \theta) \\ &- \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} \sin n(y - \theta) \end{aligned} \right\}. \end{aligned} \quad (13)$$

Again, for brevity, put

$$P = \frac{r_0^2 u_0}{a\pi^2 \sqrt{\pi t}},$$

$$f^2 = \frac{r_0^2}{4\pi^2 a^2 t};$$

and denote the difference of the two trigonometric series in the right-hand member of (13) by Q_1 . Then, the expression to be evaluated is

$$ru = P \int_0^\infty e^{-f^2 y^2} Q_1 dy. \quad (14)$$

6. We must now determine the value of Q_1 . For this purpose consider the well-known equation,

$$\frac{\frac{1}{2}(1-g^2)}{1-2g\cos\varphi+g^2} = \frac{1}{2} + g\cos\varphi + g^2\cos 2\varphi + g^3\cos 3\varphi + \dots$$

In this, for φ , substitute $\pi + y + \varphi$. Then we have

$$\frac{1}{2} - \frac{\frac{1}{2}(1-g^2)}{1+2g\cos(y+\varphi)+g^2} = g\cos(y+\varphi) - g^2\cos 2(y+\varphi) + \dots$$

Multiply this by $d\varphi$ and integrate between the limits $+\theta$ and $-\theta$. The result is

$$\begin{aligned} \frac{1}{2} \int_{-\theta}^{+\theta} d\varphi - \frac{1}{2} \int_{-\theta}^{+\theta} \frac{(1-g^2) d\varphi}{1+2g \cos(y+\varphi)+g^2} = & + g \sin(y+\theta) - g \sin(y-\theta) \\ & - \frac{1}{2} g^2 \sin 2(y+\theta) + \frac{1}{2} g^2 \sin 2(y-\theta) \\ & + \frac{1}{3} g^3 \sin 3(y+\theta) - \frac{1}{3} g^3 \sin 3(y-\theta) \\ & - \dots + \dots \\ = Q, \text{ say.} \end{aligned} \quad (15)$$

Comparing the series in (13) and (14) with that in (15), it appears that $Q_1 = Q$ when $g = 1$. Hence equation (14) becomes

$$ru = \frac{1}{2} P \int_{-\theta}^{+\theta} d\varphi \int_0^{\infty} e^{-\nu^2 y^2} dy - \frac{1}{2} P \int_{-\theta}^{+\theta} \frac{(1-g^2) d\varphi}{1+2g \cos(y+\varphi)+g^2} \int_0^{\infty} e^{-\nu^2 y^2} dy. \quad (16)$$

$g = 1.$

The first term in the second member of this equation presents no difficulty. Its value is

$$\frac{P\theta\sqrt{\pi}}{2\rho}.$$

But in the second term the element-function of the integral with respect to φ is zero when $g = 1$, except when $\cos(y+\varphi) = -1$. In this case the element-function becomes

$$\frac{1-g^2}{(1-g)^2} = \frac{1+g}{1-g} = \infty \quad \text{for } g = 1.$$

The value of this integral with respect to φ is therefore the value of the single element $\frac{1+g}{1-g} d\varphi$ when $g = 1$. To find its value the most direct process appears to be the following*:—

Replace $\cos(y+\varphi)$ by its equivalent $1 - 2 \sin^2 \frac{1}{2}(y+\varphi)$. Then, after a little reduction, the integral becomes

$$\int_{-\theta}^{+\theta} \frac{\frac{1-g}{1+g} \sec^2 \frac{1}{2}(y+\varphi) d(\frac{1}{2}\varphi)}{1 + \left(\frac{1-g}{1+g}\right)^2 \tan^2 \frac{1}{2}(y+\varphi)}.$$

Now every element of this integral is zero for $g = 1$ except that for which

* The mathematical reader will recognize in the integral to be evaluated one which Poisson and other writers have used in proving the law of development of a function in a series of periodic terms. The difficulty, if it be such, occurs in another form also in the proofs of that law given by Dirichlet and others. I am not certain that the evaluation given in the text will commend itself; but it seems to be more direct than, and quite as obvious as, Poisson's process.

$y + \varphi = (2n + 1)\pi$, n being any integer or zero; and the value of the exceptional element is dependent on θ only so far as it determines within what limits y can make $y + \varphi = (2n + 1)\pi$. We have, therefore, simply to consider what value the above expression has when $y + \varphi$ varies between $(2n + 1)\pi - i$ and $(2n + 1)\pi + i$, i being an infinitesimal. The indefinite integral is

$$\arctan \left[\frac{1-g}{1+g} \tan \frac{1}{2}(y + \varphi) \right];$$

and since, when $(1-g)$ is infinitesimal, $\frac{1-g}{1+g} \tan \frac{1}{2}(y + \varphi)$ varies from $-\infty$ to $+\infty$ as $y + \varphi$ varies over its range, the proper limits of integration are $-\infty$ and $+\infty$. Hence we conclude that

$$\frac{1}{2} \int_{-\theta}^{+\theta} \frac{(1-g^2) d\varphi}{1 + 2g \cos(y + \varphi) + g^2} = \left[\arctan \left(\frac{1-g}{1+g} \tan \frac{1}{2}(y + \varphi) \right) \right]_{-\infty}^{+\infty} = \pi.$$

It remains to find what values of $y + \varphi$ will give $\cos(y + \varphi) = -1$, or what values of y will make $y + \varphi$ an odd multiple of π . Since φ may have any value between $-\theta$ and $+\theta$, the range of values of y , which make $y + \varphi$ an odd multiple of π , will be from

$$(2n + 1)\pi - \theta \quad \text{to} \quad (2n + 1)\pi + \theta,$$

n being any integer or zero. That is, y must have values lying between

$$\begin{aligned} \pi - \theta \quad & \text{and} \quad \pi + \theta, \\ 3\pi - \theta \quad & \text{and} \quad 3\pi + \theta, \\ 5\pi - \theta \quad & \text{and} \quad 5\pi + \theta, \\ & \vdots \end{aligned}$$

7. Recurring now to equation (16) and attending to the limits of y just derived we have

$$ru = \frac{P\theta\sqrt{\pi}}{2p} - P\pi \left[\int_{\pi-\theta}^{\pi+\theta} e^{-p^2 y^2} dy + \int_{3\pi-\theta}^{3\pi+\theta} e^{-p^2 y^2} dy + \dots \right].$$

Restoring the values of P , p , and θ , and replacing py by z , there results

$$ru = ru_0 - \frac{2r_0\theta_0}{1-\pi} \left[\int_{\frac{r_0-r}{2a_1/t}}^{\frac{r_0+r}{2a_1/t}} e^{-z^2} dz + \int_{\frac{3r_0-r}{2a_1/t}}^{\frac{3r_0+r}{2a_1/t}} e^{-z^2} dz + \dots \right]. \quad (17)$$

For brevity, let us put

$$\begin{aligned} m_0 &= r_0/(2a_1/t), \\ m &= r/(2a_1/t). \end{aligned} \quad (18)$$

Then observing that $\frac{2r_0u_0}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz = r_0u_0$,

we get the following equivalent forms from (17):—

$$ru = ru_0 - \frac{2r_0u_0}{\sqrt{\pi}} \left[\int_{m_0-m}^{m_0+m} e^{-z^2} dz + \int_{3m_0-m}^{3m_0+m} e^{-z^2} dz + \dots \right], \quad (19)$$

$$ru = ru_0 - r_0u_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{m_0-m} e^{-z^2} dz - \frac{2}{\sqrt{\pi}} \int_{m_0+m}^{3m_0-m} e^{-z^2} dz - \dots \right], \quad (20)$$

$$r(u_0 - u) = \frac{2r_0u_0}{\sqrt{\pi}} \left[\int_{m_0-m}^{m_0+m} e^{-z^2} dz + \int_{3m_0-m}^{3m_0+m} e^{-z^2} dz + \dots \right], \quad (21)$$

$$r(u_0 - u) = r_0u_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{m_0-m} e^{-z^2} dz - \frac{2}{\sqrt{\pi}} \int_{m_0+m}^{3m_0-m} e^{-z^2} dz - \dots \right]. \quad (22)$$

Equations (19) and (20) give the product of the radius r and the excess of temperature u of any shell of infinitesimal thickness at any time. Equations (21) and (22) give the product of the radius and the fall in temperature of the same shell during the time t .

8. It will now be well to determine whether equations (19) to (22) satisfy the conditions (1) to (5) and also to test their equivalence with (10).

Differentiating equation (20) or (22) with respect to t we find

$$\begin{aligned} \frac{\partial (ru)}{\partial t} &= \frac{2r_0u_0}{\sqrt{\pi}} e^{-(m_0-m)^2} \cdot \frac{\partial (m_0-m)}{\partial t} \\ &+ \frac{2r_0u_0}{\sqrt{\pi}} e^{-(3m_0-m)^2} \cdot \frac{\partial (3m_0-m)}{\partial t} \\ &- \frac{2r_0u_0}{\sqrt{\pi}} e^{-(m_0+m)^2} \cdot \frac{\partial (m_0+m)}{\partial t} \\ &+ \dots \end{aligned}$$

But, from (18),

$$m_0 - m = \frac{r_0 - r}{2a} t^{-\frac{1}{2}},$$

$$m_0 + m = \frac{r_0 + r}{2a} t^{-\frac{1}{2}},$$

...

Hence
$$\frac{\partial(m_0 - m)}{\partial t} = \frac{r_0 - r}{4a} t^{-\frac{3}{2}} = -\frac{1}{2}(m_0 - m) t^{-1},$$

$$\frac{\partial(m_0 + m)}{\partial t} = \frac{r_0 + r}{4a} t^{-\frac{3}{2}} = -\frac{1}{2}(m_0 + m) t^{-1},$$

$$\dots \dots \dots ;$$

and
$$\frac{\partial(ru)}{\partial t} = -\frac{r_0 u_0}{t\sqrt{\pi}} [(m_0 - m) e^{-(m_0 - m)^2} - (m_0 + m) e^{-(m_0 + m)^2} + (3m_0 - m) e^{-(3m_0 - m)^2} + \dots]. \quad (23)$$

Differentiating (20) with respect to r , there results,

$$\frac{\partial(ru)}{\partial r} = \frac{2r_0 u_0}{\sqrt{\pi}} \left\{ \begin{aligned} &e^{-(m_0 - m)^2} \cdot \frac{\partial(m_0 - m)}{\partial r} \\ &+ e^{-(3m_0 - m)^2} \cdot \frac{\partial(3m_0 - m)}{\partial r} \\ &- e^{-(m_0 + m)^2} \cdot \frac{\partial(m_0 + m)}{\partial r} \\ &+ \dots \dots \dots \end{aligned} \right\}.$$

By means of (18)
$$\frac{\partial(m_0 - m)}{\partial r} = -\frac{1}{2a\sqrt{t}},$$

$$\frac{\partial(3m_0 - m)}{\partial r} = -\frac{1}{2a\sqrt{t}},$$

$$\frac{\partial(m_0 + m)}{\partial r} = +\frac{1}{2a\sqrt{t}},$$

$$\dots \dots \dots$$

Therefore

$$\frac{\partial(ru)}{\partial r} = u_0 - \frac{r_0 u_0}{a\sqrt{\pi t}} [e^{-(m_0 - m)^2} + e^{-(m_0 + m)^2} + e^{-(3m_0 - m)^2} + \dots]. \quad (24)$$

Differentiating the last equation again, we find

$$\frac{\partial^2(ru)}{\partial r^2} = -\frac{r_0 u_0}{a^2 t \sqrt{\pi}} \left\{ \begin{aligned} &(m_0 - m) e^{-(m_0 - m)^2} \\ &- (m_0 + m) e^{-(m_0 + m)^2} \\ &+ (3m_0 - m) e^{-(3m_0 - m)^2} \\ &- \dots \dots \dots \end{aligned} \right\}. \quad (25)$$

Dividing (23) by (25), we get
$$\frac{\partial(ru)}{\partial t} = a^2 \frac{\partial^2(ru)}{\partial r^2},$$

which is equation (1). The equations (19) to (24) therefore fulfill the first requirement.

If we make $r = r_0$, equation (19) becomes

$$\begin{aligned} r_0 u &= r_0 u_0 - \frac{2r_0 u_0}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz \\ &= r_0 u_0 - r_0 u_0 = 0, \end{aligned}$$

and hence $u = 0$. Condition (2) is thus satisfied.

If we make $r = 0$, equation (20) gives

$$ru = 0 - r_0 u_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz \right) = 0,$$

as required by condition (3).

When $t = 0$, equation (20) becomes

$$\begin{aligned} ru &= ru_0 - r_0 u_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz \right) \\ &= ru_0, \end{aligned}$$

or

$$u = u_0,$$

which is in accordance with condition (4).

Finally, equations (19) to (22) should satisfy condition (5). But if we make $t = \infty$ in (21), the result is

$$r(u_0 - u) = \frac{2r_0 u_0}{\sqrt{\pi}} \left(\int_0^0 e^{-z^2} dz + \int_0^0 e^{-z^2} dz + \dots \right),$$

which is ambiguous. The same ambiguity appertains to equation (22). This difficulty will appear also when we seek to determine the temperature at the centre of the sphere after an indefinitely great, or infinite, time. But it will be observed that if we can derive an expression from equations (19) to (22) for the temperature at the centre of the sphere, and show this expression to be zero when $t = \infty$, then those equations must satisfy (5).

If we make $r = 0$ in (20), it becomes

$$u = u_0 - r_0 u_0 \frac{1 - 1}{0}.$$

But the true value of this expression is readily found by differentiation to be

$$u = u_0 \left[1 - \frac{4m_0}{\sqrt{\pi}} (e^{-m_0^2} + e^{-9m_0^2} + e^{-25m_0^2} + \dots) \right]. \quad (26)$$

$$r = 0.$$

Now since $m_0 = r/2\alpha_1/t$, the last equation assumes the indeterminate form when

$t = \infty$. But from the Eulerian integral of § 4 we have

$$e^{-n^2 m_0^2} = e^{-n^2 / m_0^{-2}} = \frac{1}{m_0 \pi} \int_0^\infty e^{-(\frac{1}{2}x / m_0)^2} dx \cos nx.$$

Making n successively equal to 1, 2, 3, . . . in this expression, and substituting the equivalents in (26), we get

$$u = u_0 \left[1 - \frac{4}{\pi} \int_0^\infty e^{-(\frac{1}{2}x / m_0)^2} dx (\cos x + \cos 3x + \cos 5x + \dots) \right].$$

By reference to the series used in § 6 it is seen that when $g = 1$,

$$\frac{\frac{1}{4}(1-g^2)}{1-2g \cos x + g^2} - \frac{\frac{1}{4}(1-g^2)}{1+2g \cos x + g^2} = \cos x + \cos 3x + \cos 5x + \dots$$

Hence we have when $g = 1$,

$$u = u_0 \left[1 - \frac{4}{\pi} \int_0^\infty e^{-(\frac{1}{2}x / m_0)^2} dx \left(\frac{\frac{1}{4}(1-g^2)}{1-2g \cos x + g^2} - \frac{\frac{1}{4}(1-g^2)}{1+2g \cos x + g^2} \right) \right]. \quad (27)$$

When t is infinite, m_0 is infinitesimal, and we may for this value of t restrict the upper limit in (27) to an infinitesimal i , since for all greater values of x , $e^{-(\frac{1}{2}x / m_0)^2}$ vanishes. In this case, also, the second term under the sign of integration in (27) may be dropped, by reason of the infinitesimal factor $(1-g^2)$. Therefore, when $t = \infty$ and $g = 1$, (27) becomes

$$u = u_0 \left[1 - \frac{1}{\pi} \int_{-i}^{+i} \frac{e^{-(\frac{1}{2}x / m_0)^2} (1-g^2) d(\frac{1}{2}x)}{1-2g \cos x + g^2} \right]. \quad (28)$$

Since i may be assumed to be of a lower order than m_0 , the value of the factor $e^{-(\frac{1}{2}x / m_0)^2}$ within the limits of integration is unity. The value of the integral then, as shown in § 6, is π . Therefore we conclude that when $r = 0$ and $t = \infty$,

$$u = u_0 [1 - (1/\pi)\pi] = 0,$$

and hence that equations (19) to (22) satisfy condition (5).

9. Having established the complete equivalence of equations (19) to (22) and (10), we may now consider some practical features pertaining to their applications. In the first place it will be observed that the definite integrals in either of equations (19) to (22) form an extremely converging series for all but very great values of the time. So converging are these series that, for the case of the earth, the first term will suffice when t is as great 100,000,000,000 years. When t exceeds this time, recourse may be had to the original series (10), which then becomes highly converging also, as the following numerical examples will show.

10. If we confine the time to relatively small values, it will suffice to use the

first term of (19) to (20). If, further, we restrict r to values which do not differ sensibly from r_0 , we may write (22) in the form

$$u_0 - u = u_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{r_0 - r}{2a\sqrt{t}}} e^{-z^2} dz \right). \quad (29)$$

This agrees with an expression which Sir William Thomson has used for the fall of temperature, in his paper, *On the Secular Cooling of the Earth*.*

11. To illustrate the numerical application of the formulæ derived, it will be well to assume such a value of the time that for the earth both sets of equations, i. e. (10) and (11), and (19) to (24), and (26) may be applied.

As a first example let

$$r = \frac{1}{2}r_0 \quad \text{and} \quad a^2(\pi/r_0)^2 t = 1.$$

The value of t required in the case of the earth to give this last equality is, as we have seen in §3, about one hundred thousand million years. With these data equation (10) gives

$$u = \frac{4u_0}{\pi} (e^{-1} - \frac{1}{3}e^{-9} + \dots).$$

| | log | No. |
|--|----------------------------|-------------|
| e^{-1} | 9.5657055 - 10 | + 0.3678794 |
| e^{-9} | 6.09135 - 10 $\frac{1}{3}$ | - 0.0000411 |
| $(e^{-1} - \frac{1}{3}e^{-9} + \dots)$ | 9.5656570 - 10 | 0.3678383 |
| 4 | 0.6020600 | |
| π | 0.4971499 | |
| u/u_0 | 9.6705671 - 10 | 0.4683463 ; |
| $u =$ | 0.4683463 u_0 . | |

hence

* Thomson and Tait's *Natural Philosophy*, Vol. I. Part II. Appendix (D).

The law of cooling used by Thomson in this paper is that which applies to the diffusion of temperature in an infinite solid on the supposition that at the beginning of the time the temperature had two different constant values on the two sides of an infinite plane. The formula which expresses this law is

$$v = v_0 + \frac{2V}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-z^2} dz,$$

in which v_0 is the half sum and V the half difference of the initial temperatures; x is the distance of any point whose temperature is v , from the initial plane, the sign of x being positive for the hotter side and negative for the cooler side of the plane; t is the time from the initial epoch and k is the same as our a^2 .

Now when $t = 0$ the formula gives $v = v_0 + V$, and hence the fall of temperature in any time t is

$$v_0 + V - \left[v_0 + \frac{2V}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-z^2} dz \right] = V \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-z^2} dz \right].$$

This agrees with (29) if we make $V = u_0$ and $x = r_0 - r$.

Again, when $a^2(\pi/r_0)^2 t = 1$, $2a_1/t = 2r_0/\pi$; and when $r = \frac{1}{2}r_0$, equations (18) give

$$\begin{aligned} m_0 &= \frac{1}{2}\pi, \\ m &= \frac{1}{4}\pi, \\ m_0 - m &= \frac{1}{4}\pi = 0.7853982, \\ m_0 + m &= \frac{3}{4}\pi = 2.3561945, \\ 3m_0 - m &= \frac{5}{4}\pi = 3.9269908, \\ 3m_0 + m &= \frac{7}{4}\pi = 5.4977871. \end{aligned}$$

From the tables of the integral $\int_0^z e^{-z^2} dz$ given in Oppolzer's *Lehrbuch zur Bahnbestimmung*, Vol. II. Table X, we find for the first two terms in (21)

$$\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} e^{-z^2} dz = 0.2355828.78,$$

$$\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} e^{-z^2} dz = 0.0000000.25.$$

The sum of these integrals is

| | |
|-------------------------|---------------------------|
| | log |
| 0.2355829 | 9.3721438 — 10, |
| 4. | 0.6020600, |
| $1/\pi$ | 0.2485749, |
| $\frac{u_0 - u}{u_0}$, | 0.5316538 9.7256289 — 10. |

Hence by (21)

$$u_0 - u = 0.5316538 u_0,$$

or

$$u = 0.4683462 u_0.$$

This agrees with the value derived above from (10) within one unit in the seventh place of decimals. It will be observed that the contribution of the second term in (21) is less than one unit in the seventh place. The first term of (21) would therefore suffice when the time is much greater than 100,000,000,000 years for all practical computations relative to the temperature of points within the earth.

12. As a second illustration, let us compute the temperature at the centre of the sphere when m_0 of (18) and (26) has the value

$$m_0 = \frac{r_0}{2a_1/t} = 1.$$

This gives

$$t = \frac{r_0^2}{4a^2},$$

which for the case of the earth gives $t = 273,000,000,000$ years. We also have for use in (11), when $m_0 = 1$,

$$a^2 \left(\frac{\pi}{r_0} \right) = \frac{\pi^2}{4}.$$

Hence the computation by (11) runs thus:—

| | No. | log |
|-------------|-------------------------------|---------------------------|
| 1st term. | $\log e$ | 9.6377843 — 10 |
| | π^2 | 0.9942997 |
| | 4 | 0.6020600 |
| | $\log e^{(\frac{1}{2}\pi)^2}$ | 0.0300240 |
| | $e^{(\frac{1}{2}\pi)^2}$ | 1.0715785 |
| + 0.0848050 | $e^{-(\frac{1}{2}\pi)^2}$ | 8.9284215 — 10. |
| 2d term. | $\log e^{(\frac{1}{2}\pi)^2}$ | 0.03002 |
| | 4 | 0.60206 |
| | $\log e^{\pi^2}$ | 0.63208 |
| | e^{π^2} | 4.2863 |
| — 0.0000517 | $e^{-\pi^2}$ | 5.7137 _n — 10. |

The third and higher terms are insignificant. The sum of the first two is 0.0847533. Hence by (11)

$$u = 0.1695066 u_0.$$

According to (26) the computation is as follows:—

| | log |
|--|-----------------|
| + 0.3678794 e^{-1} | 9.5657055 — 10 |
| + 0.0001234 e^{-9} | 6.09135 — 10 |
| + 0.3680028 $e^{-1} + e^{-9} + \dots$ | 9.5658511 — 10 |
| 4 | 0.6020600 |
| $1/\pi$ | 0.2485749 |
| 0.8304935 $(4/\pi)(e^{-1} + e^{-9} + \dots)$ | 9.9193362 — 10. |

Therefore by (26) $u = u_0(1 - 0.8304935)$
 $= 0.1695065 u_0.$

Thus the results derived from (11) and (26) for the temperature at the centre of the sphere agree within the limits of precision attainable with 7-place logarithms.

SOLUTIONS OF EXERCISES.

32

A TRIANGLE PQR is inscribed in the triangle ABC . Determine the ratios in which P, Q, R divide the sides BC, CA, AB in order that AQR, BRP, CPQ may be respectively $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ of ABC . [W. M. Thornton.]

SOLUTION.

Put $BP = ax, CQ = by, AR = cz$. Then the hypothesis gives

$$(1 - y)z = \frac{1}{3},$$

$$(1 - z)x = \frac{1}{3},$$

$$(1 - x)y = \frac{1}{3}.$$

Elimination gives $35x^2 - 38x + 9 = 0$.

Whence we find x , and by substitution, y and z . The results are

$$x = \frac{19 + \sqrt{46}}{35}, \quad y = \frac{16 + \sqrt{46}}{42}, \quad z = \frac{26 + \sqrt{46}}{45};$$

in which either value of $\sqrt{46}$ may be taken. [H. N. Draughton.]

82

THE major axes of two similar and equal concentric ellipses intersect at right angles, and the area common to the two curves is half that of either ellipse. Find the eccentricity. [Ormond Stone.]

SOLUTION.

Let $ACB, A'CB'$ be quadrants of the two ellipses, the curves intersecting in D . Join CD , and draw DMN parallel to AC , intersecting the circle drawn about C with radius $CB = CB'$ in M , and CB in N , and join CM . The circular sector CBM is formed from the elliptic sector CBD , by reducing the breadths parallel to CA in the ratio b/a . Hence the area of CBM is the same part of the whole circle that CBD is of the whole ellipse. But by hypothesis, CBD is $\frac{1}{16}$ of the whole ellipse. Hence CBM is $\frac{1}{16}$ of the whole circle, and the angle MCB is therefore equal to $\frac{1}{8}\pi$. Or since by hypothesis $DN = CN$,

$$MN : DN = b : a;$$

$$\therefore b/a = \tan \frac{1}{8}\pi;$$

$$\therefore e^2 = 1 - \tan^2 \frac{1}{8}\pi = 2\sqrt{2} - 2. \quad [O. L. Mathiot.]$$

107

GIVEN one vertex of a rectangle and the ratio of its sides, construct the rect-

angle so that the extremities of the diagonal opposite to the given vertex shall lie

1. On two given parallel straight lines,
2. On two given intersecting straight lines,
3. One on a given line and the other on a given circle,
4. On two given circles.

[*R. D. Bohannon.*]

SOLUTION.

Let C be the given vertex; AB the required diagonal; and α, β the lines on which A, B are to lie. When A describes α , A' , the extremity of $CA' = n \cdot CA$, will describe a figure α' , similar and similarly placed to α and n times as large. B , the extremity of $CB = CA'$, will describe an equal figure β' , located by turning α' through a right angle about C . The points of intersection of β' with β furnish solutions of the problem. The application of the general method to case (4) of the problem is easy. In (4) β' is a circle cutting β in two points, each of which gives a solution.

[*W. M. Thornton.*]

[Mr. Henry Heaton sends substantially the same solution. Mr. E. L. Stabler suggests the generalized form applicable to other curves.]

108

CONSTRUCT an equilateral triangle of given size inscribed in a given equilateral hyperbola.

[*R. H. Graves.*]

SOLUTION.

Any rectangular hyperbola, passing through the vertices of a triangle, passes through its orthocentre. Also the nine-point circle is the locus of the centres of rectangular hyperbolas passing through the vertices of a triangle. Therefore the centre of the required triangle lies on the given hyperbola, and its inscribed circle passes through the centre of the curve. Hence the following construction:—

Construct a diameter equal to two-thirds of the altitude of the triangle, with one end of the diameter as a centre and a radius equal to this diameter describe a circle; three of the points of intersection of hyperbola and circle are the vertices of the required triangle. (Omit point of intersection at end of diameter, for otherwise the diameter would bisect a chord perpendicularly, which is, *in general*, impossible.) If two-thirds of the altitude is less than the transverse axis, the construction is obviously impossible.

[*R. H. Graves.*]

110

SOLVE the triangle ABC ; given a, A and the product $\beta\gamma$ of the bisectrices drawn from B, C to CA, AB .

SOLUTION.

With the customary notations we get from the triangles BCE, BCF

$$\frac{\beta}{a} = \frac{\sin C}{\sin(A + \frac{1}{2}B)}, \quad \frac{\gamma}{a} = \frac{\sin B}{\sin(A + \frac{1}{2}C)},$$

and therefore

$$\frac{\beta\gamma}{a^2} = \frac{\sin B \sin C}{\sin(A + \frac{1}{2}B) \cdot \sin(A + \frac{1}{2}C)},$$

$$\frac{\beta\gamma}{2a^2} = \frac{\cos^2 \frac{1}{2}(B - C) - \cos^2 \frac{1}{2}(B + C)}{\cos \frac{1}{2}(B - C) + \sin \frac{3}{2}A}.$$

Conversely from this relation $\frac{1}{2}(B - C)$ can be computed, and as $\frac{1}{2}(B + C)$ is also known, B and C , and from them the other parts of the triangle, are determined.

[Henry Heaton; T. U. Taylor.]

112

GENERALIZATION.

To inscribe in the given triangle ABC a triangle PQR whose sides QR , RP , PQ make given angles with BC , CA , AB .

SOLUTION I.

The conditions of the problem give the directions of the sides of PQR . Draw a parallel to RP , intersecting AB in P' and CA in R' . Through P' and R' draw parallels to QR and PQ , intersecting at Q' . Through Q' draw a parallel to BC , intersecting CA at C' . Through C draw a parallel to $C'P'$. This parallel will intersect AB at P . The remainder of the solution follows immediately.

[A. B. Evans; E. L. Stabler; Ormond Stone.]

SOLUTION II.

Draw the triangle $P'Q'R'$ whose sides $Q'R'$, $R'P'$, $P'Q'$ pass through A , B , C and are parallel to QR , RP , PQ , and another $A'B'C'$ whose sides $B'C'$, $C'A'$, $A'B'$ pass through P' , Q' , R' and are parallel to BC , CA , AB . Divide the sides of ABC at P , Q , R in the same ratios in which the corresponding sides of $A'B'C'$ are divided by P' , Q' , R' . PQR is the triangle sought.

[Sarah Szold.]

113

THE transversal MN meets the sides AB , AC of the fixed triangle ABC , and makes

$$AM \cdot AN = BM \cdot CN.$$

Find its envelope.

[O. Root, Jr.]

SOLUTION I.

According to the given condition $\left(\frac{AM}{BM} = \frac{CN}{AN}\right)$, the point M divides the segment AB in the same ratio in which N divides the segment CA ; in other words, the variable points M and N form similar projective ranges on the fixed lines AB , CA . Hence, the variable line MN which connects corresponding points of these ranges envelopes a parabola. (See, for instance, L. Cremona's *Projective Geometry*, translated by Leudesdorf, Oxford, 1885, p. 128.) The lines AB and AC are tangent to this parabola in B and C respectively, these being the

points corresponding to the point of intersection A of the two ranges.

[*Alexander Ziwet.*]

SOLUTION II.

Referring all parts of the figure to AB, AC as co-ordinate axes, and putting

$$AM = m, \quad AN = n, \quad AB = b, \quad AC = c,$$

we have for the equation to MN

$$\frac{x}{m} + \frac{y}{n} = 1,$$

and the parameters m, n are connected by the relation,

$$\frac{m}{b} + \frac{n}{c} = 1.$$

Differentiation and elimination give

$$\frac{m^2}{bx} = \frac{n^2}{cy} = \sqrt{\frac{x}{b}} + \sqrt{\frac{y}{c}};$$

whence for the equation to the enveloping parabola we have

$$\sqrt{\frac{x}{b}} + \sqrt{\frac{y}{c}} = 1. \quad [W. M. Thornton.]$$

114

A CIRCLE intersects a conic in four points, P_1, P_2, P_3, P_4 . Show that if the x -axis be parallel to the axis of the conic the area of their quadrilateral is

$$(x_2 - x_4)(y_1 - y_3). \quad [R. H. Graves.]$$

SOLUTION.

Let the x -axis be parallel to, or coincident with, the axis of the conic, and let the co-ordinates of the vertices taken in order be $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$. Let θ be the acute angle that each diagonal makes with the x -axis.* Then the area of the quadrilateral = $\frac{1}{2}$ product of its diagonals into $\sin 2\theta$

$$= \pm \frac{1}{2} \frac{x_2 - x_4}{\cos \theta} \cdot \frac{y_1 - y_3}{\sin \theta} \sin 2\theta = \pm (x_2 - x_4)(y_1 - y_3).$$

The same formula may be proved otherwise for the parabola as follows:—

Let $y^2 = 4ax$ be the equation to the parabola. Then

$$\text{Area} = \pm \frac{1}{2} \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} = \pm \frac{1}{8a} \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \end{vmatrix}$$

* Salmon's *Conic Sections*, 244.—[*W. M. T.*]

$$\begin{aligned}
&= \pm \frac{1}{8a} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ y_1 - y_3 & y_2 - y_4 & y_3 & y_4 \\ y_1^2 - y_3^2 & y_2^2 - y_4^2 & y_3^2 & y_4^2 \end{vmatrix} = \pm \frac{1}{8a} (y_1 - y_3)(y_2 - y_4) \begin{vmatrix} 1 & 1 \\ y_1 + y_3 & y_2 + y_4 \end{vmatrix} \\
&= \pm \frac{1}{8a} \begin{vmatrix} 2 & 1 \\ y_1 + y_2 + y_3 + y_4 & y_2 + y_4 \end{vmatrix} (y_1 - y_3)(y_2 - y_4) \\
&= \pm \frac{1}{4a} (y_1 - y_3)(y_2 - y_4) \begin{vmatrix} 1 & 1 \\ 0 & y_2 + y_4 \end{vmatrix} \\
&= \pm \frac{1}{4a} (y_1 - y_3)(y_2^2 - y_4^2) = \pm (x_2 - x_4)(y_1 - y_3). \quad [R. H. Graves.]
\end{aligned}$$

117

SHOW that the angle between the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the tangent at the corresponding point of the principal circle is given by the formula,

$$\tan \delta = \frac{a \tan \varphi}{(1 - a) + \tan^2 \varphi},$$

where $a = (a - b)/a$ is the ellipticity, and φ is the eccentric anomaly.

SOLUTION.

Let (x', y') be the co-ordinates of the point of tangency with the principal circle; (x', y'') those of the corresponding point of the ellipse. The equations of the tangents to the circle and the ellipse are

$$y' = -\frac{x'}{y'}x + \frac{a^2}{y'},$$

and

$$y' = -\frac{b^2x'}{a^2y''}x + \frac{b^2}{y''}.$$

Hence the tangents p, q of the angles which the tangents to the circle and the ellipse make with the axis of x , are

$$p = -\frac{x'}{y'}, \quad q = -\frac{b^2x'}{a^2y''}.$$

But

$$x' = a \cos \varphi, \quad y' = a \sin \varphi, \quad y'' = b \sin \varphi;$$

$$\therefore p = -\frac{1}{\tan \varphi}, \quad \text{and} \quad q = -\frac{b}{a \tan \varphi}.$$

Then

$$\tan \delta = \frac{q - p}{1 + pq} = \frac{a \tan \varphi}{(1 - a) + \tan^2 \varphi}.$$

[William Hoover; C. D. Schmitt.]

118

SHOW that the eccentric anomaly which gives the maximum deviation is $\tan^{-1} \sqrt{1-a}$, and find the co-ordinates of the point of contact T and the equation to the tangent t .

SOLUTION.

The deviation will be a maximum when $\tan \delta$ is a maximum. Equating the first differential coefficient to zero,

$$a[(1-a) + \tan^2 \varphi] \sec^2 \varphi - a \tan \varphi \cdot 2 \tan \varphi \sec^2 \varphi = 0;$$

whence

$$\varphi = \tan^{-1} \sqrt{1-a};$$

$$\therefore \sin \varphi = \sqrt{\frac{1-a}{2-a}}, \quad \cos \varphi = \frac{1}{\sqrt{2-a}},$$

and

$$x' = \frac{a}{\sqrt{2-a}}, \quad y'' = \frac{a(1-a)^{\frac{3}{2}}}{\sqrt{2-a}}.$$

The equation of the tangent t is therefore

$$\frac{a^3(1-a)^{\frac{3}{2}}}{1(2-a)} y + \frac{ab^2}{\sqrt{2-a}} x = a^2 b^2. \quad [\text{William Hoover.}]$$

119

SHOW that the portion of t intercepted between the axes equals in length the sum of the semi-axes a, b , and is divided at T into these two parts.

SOLUTION.

Let $x=0$ and $y=0$ in order in the equation of the tangent. The corresponding intercepts are $y_1 = \frac{b^2}{a} \cdot \frac{\sqrt{2-a}}{(1-a)^{\frac{3}{2}}}$, $x_1 = a \sqrt{2-a}$. Then $t = \sqrt{x_1^2 + y_1^2} = a + b$. If s be the segment adjacent to the axis of x , $s : y'' :: t : y_1$, whence $s = b$; and the other is a . [William Hoover.]

120

SHOW that the circumcircle of txy passes through the centre of curvature at T , and that the area of the circle of curvature equals that of the ellipse.

SOLUTION.

If ρ = the radius of curvature, we have for the ellipse,

$$\rho = \frac{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^{\frac{3}{2}}}{ab} = \sqrt{ab},$$

by using $\sin \varphi, \cos \varphi$ given in solution of 118. The radius of the circumcircle of $txy = \frac{1}{2}(a+b)$, and the distance of its centre from $T = \frac{1}{2}(a-b)$. \therefore since $\rho, \frac{1}{2}(a+b), \frac{1}{2}(a-b)$ are the sides of a right triangle, the circumcircle passes through the centre of curvature T . We have $\pi \rho^2 = \pi ab$. [William Hoover.]

121

SHOW that when the ellipse varies subject to the condition, $a + b$ constant, the locus of T is the hypocycloid,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a + b)^{\frac{2}{3}},$$

which touches each ellipse at the point T .

SOLUTION.

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (1)

and we have $a + b = c = \text{a constant}$. (2)

Differentiating (1) and (2), a and b only variable,

$$\frac{x^2}{a^2} da = -\frac{ay^2}{b^3} db, \quad (3)$$

$$da = -db; \quad (4)$$

$$\therefore \frac{x^2}{a^2} = \frac{ay^2}{b^3}. \quad (5)$$

$$\therefore (1) \text{ is } \frac{y^2}{b^3}(a + b) = 1, \text{ or } b = c^{\frac{1}{3}}y^{\frac{1}{3}}, \text{ and } a = c^{\frac{1}{3}}x^{\frac{1}{3}}.$$

These in (2) give $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a + b)^{\frac{2}{3}}$, a hypocycloid. From this equation $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$, which for $x = x' = \frac{a}{1 - a}$, $y = y'' = \frac{a(1 - a)}{1/(2 - a)}$, becomes $-1/(1 - a)$, showing that the ellipse and hypocycloid have a common tangent at T . [William Hoover.]

EXERCISES.

137

INTEGRATE the differential,

$$\frac{\sin \theta \cos \theta}{\sin^4 \theta + \cos^4 \theta} d\theta. \quad [O. Root, Jr.]$$

138

ABOUT the vertices of an equilateral triangle three spheres are drawn with radii equal to the side of the triangle. Find the volume common to them all.

[W. M. Thornton.]

139

AN ellipse touches the given ellipse $a^2y^2 + b^2x^2 = a^2b^2$ at the extremities of a diameter and has its foci on the curve. Find the position of its axis major, and the position of the foci when the angle between the given diameter and the axis major of the given ellipse is a maximum. [R. H. Graves.]

140

IN the triangle ABC , the radius of the inscribed circle is

$$\frac{1}{3} \frac{(b+c) \cos^2 \frac{1}{2}A + (c+a) \cos^2 \frac{1}{2}B + (a+b) \cos^2 \frac{1}{2}C}{\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C}.$$

[Ormond Stone.]

141

SOLVE the differential equation,

$$ud^2y + 2 du dy = v dx dy,$$

u, v, y being all functions of x .

[William Hoover.]

142

LET p be a prime number of the form $4n - 1$, n being an integer. Let the primitive p^{th} roots of unity be

$$\omega, \omega^\lambda, \omega^{\lambda^2}, \dots, \omega^{\lambda^{p-2}}.$$

Put $\theta = 2\pi/p$, and let the negative terms of the series,

$$\sin \theta, \sin(\lambda^2\theta), \sin(\lambda^4\theta), \dots, \sin(\lambda^{p-3}\theta),$$

be $\sigma', \sigma'', \sigma''', \dots$. Then

$$p = [\tan \frac{1}{4}(\rho + 1)\theta - 4(\sigma' + \sigma'' + \sigma''' + \dots)]^2.$$

Verify this relation for the particular cases $p = 3, 7, 11$, and give a general proof.

[George Paxton Young.]

143

IF a curve of the fourth order have a singular point at which are three coincident tangents, this line passes through the intersection point of the bitangent with the line of junction of the two points of inflexion of the curve.

[F. H. Loud.]

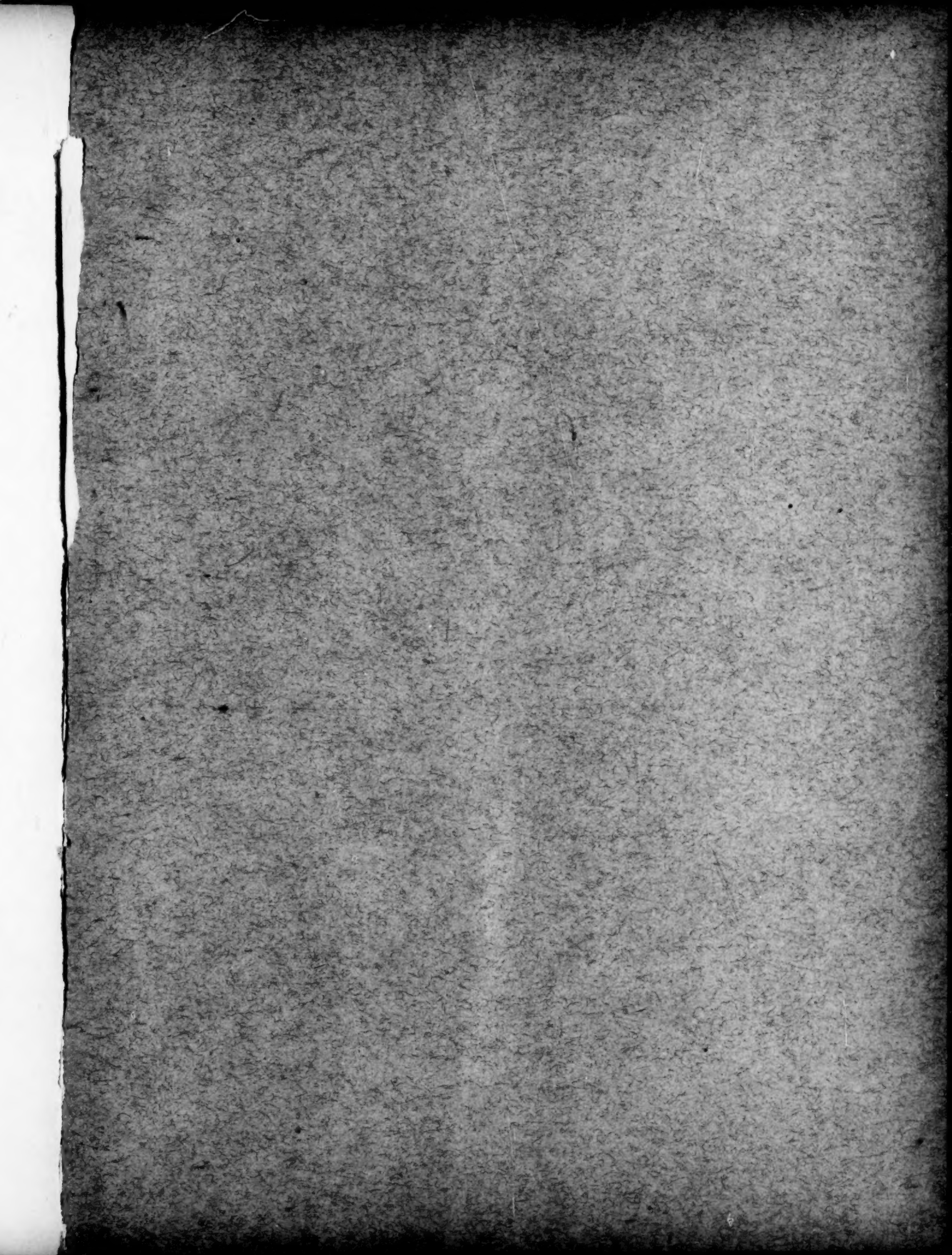
144

SHOW that the diurnal path of the shadow of the top of a vertical rod on a horizontal plane will be a rectangular hyperbola, if

$$2 \sin \delta = \sin \left(\frac{\pi}{4} + \varphi \right) + \sin \left(\frac{\pi}{4} - \varphi \right),$$

where φ is the latitude of the place and δ the declination of the sun, assumed to be constant.

[R. H. Graves.]



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